## Multiplication rule

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| If an operation consists of k steps and for any i from 1 to k, the $i^{th}$ step can be performed in $n_i$ ways, then the whole operation can be performed in $\prod_{i=1}^{k} n_i$ ways. |

P1: How many numbers can be expressed with 8 bits?

P2: How many elements are in the Cartesian product A1×A2×A3? where A1 = {2,3}, A2={a,b,c}, A3={T,F}

P3: How many possible Canadian postal codes are there?

P4: If a student council consists of 3 freshmen, 4 sophomores, 3 juniors, and 5 seniors. How many possible subcommittees of 4 people consisting of 1 representatives from each year are there?

P5: How many times will *stuff* be executed in this code fragment:

```
for (i=1; i≤100; i++)
   for (j=1; j≤20; j++)
        for (k=1; k≤5; k++)
              stuff
```

P6: How many times will *stuff* be executed in this code fragment:

```
for (i=1; i≤10; i++)
   for (j=i; j≤10; j++)
        stuff
```

## Pigeonhole Principle

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| A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least 2 elements in the domain that have the same image in the co-domain. |

P7: How many people do you need to put in a room to make sure that:
- 2 are born the same month?
- 2 are born on the same day of the year?

P8: You have a drawer with identical socks, some black and some white. How many socks do you need to pull out in the dark to be sure to get 2 of the same colour?

P9: Pairs of integers with same sum. $A=\{1,2,3,4,5,6,7,8\}$ How many integers do you need to pull out of A to be sure that at least one pair of them has a sum of 9?

## Generalized Pigeonhole Principle

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| For any function f from a finite set X to a finite set Y, and for any positive integer k,<br>if $N(X) > k.N(Y)$,<br>then $\exists y \in Y$ s.t. y is the image of at least k+1 distinct elements of X. |

P10: In a group of 100 people, how many **must** have the same middle initial?

P11: A college class has 80 students. All the students are between 18 and 35 years of age. You want to make a bet that at least x people have the same age. How big can you make x and still **be sure** to win?

### Permutations

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| For any integer $n \geq 1$, the number of permutations of a set with n elements is n! |

P12: If you have the letters D, Y, S, T, U in scrabble, in how many ways can they be combined to form a 5-letter word (meaningful or not)?

P13: In how many different ways can 0 letters be arranged?

P14: 8 runners are competing in the Olympic finals for the 100m race.
- How many possible lane configurations are there?
- How many possible outcomes will there be on the podium?

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| An r-permutation of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $P(n,r) = \frac{n!}{(n-r)!}$ |

P15: If 10 people arrive at the last minute at the theater and there are only 3 seats left: GG12, KK24, and KK25, how many possible seating arrangements will there be to fill up the theatre?

### Combinations

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| Let n and r be non-negative integers with $r \leq n$.<br>An r-combination of a set of n elements is a subset of r of the n elements. The symbol $\binom{n}{r}$ which is read "n choose r" denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.<br>$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$ |

P16: How many combinations of 3 letters can you pick from the letters of the word STUDY?

P17: Suppose that Angela, Dwayne, Hassan, Chris, Tran, and Lila are on the course union executive and 4 of them have to sit on a special committee. What are all the possible teams to do so?

## Properties of Combinations

P18: How many combinations of 1 letter can you pick from the letters of the word STUDY?

$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \text{n}$

P19: How many combinations of 5 letters can you pick from the letters of the word STUDY?

$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = \frac{n!}{n!} = 1$

P20: You are playing a card game where you only keep 5 cards in your hand during the game. However, you will pick up 3 cards at the beginning of each of your turns, and you will also discard 3 by the end of your turn. Assuming that you are playing from a single deck (i.e. all the cards are different) how many different discards can you make during your turn?

Let n and r be non-negative integers with r≤n. Then $\binom{n}{r} = \binom{n}{n-r}$

Corollary: $\binom{n}{n-1} = \binom{n}{1} = \text{n}$

Pascal's Formula:

$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

P21: What are the coefficients of $(a+b)^0$, $(a+b)^1$, $(a+b)^2$, $(a+b)^3$ ?

Binomial Theorem

Given any real numbers a and b and any non-negative integer n,

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

### Combinations with additional constraints

P22: Suppose that you need to pick a team of 5 people out of a bigger group of 12 but 2 of them always need to work together. What are all the possible combinations?

P23: Suppose that you need to pick a team of 5 people out of a bigger group of 12 but 2 of them don't want to work together. What are all the possible combinations?

P24: Representational issues: a faculty student union has 5 elected representatives from each of the 4 programs in the faculty. In addition, there are 4 elected officials (1 president and 3 vice-presidents) which can come from any program.

You would like to pick a team consisting of the president ex-officio, 1 of the vice presidents and 2 representatives from each of the program. How many possible teams are there?

### Permutations of a set with repeated elements

P25: How many permutations of the word MISSISSIPPI are there?

> If a collection consists of n objects grouped into k categories such that all the objects in the same category are indistinguishable from each other and the i^th category has $n_i$ elements. Then the number of distinct permutations of the n objects is $\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\ldots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_{k3}} = \frac{n!}{n_1!n_2!\ldots n_k!}$

P26: 8 runners are competing in the Olympic finals for the 100m race. 3 are from Jamaica, 2 from the US, and one from each of Canada, Cote d'Ivoire, and South Africa. How many lane configurations of the countries are there?

## Combinations with Repetitions Allowed

P27: You have to buy 3 boxes of cereal. The grocery store has 4 different types, all the same size and same price. It is also well stocked with 10 boxes of each type. What are all the possible types of purchases you could make?

> An r-combination with repetition allowed, or multiset of size r, chosen from a set $X=\{x_1, \ldots, x_n\}$ of n elements is an unordered selection of elements taken from X with repetition allowed. This is denoted by $[x_{i1}, \ldots, x_{ir}]$ where each $x_{ij}$ is in X and some of the $x_{ij}$ may equal each other.
>
> The number of multisets of size r selected from a set of n elements is $\binom{r+n-1}{r}$

P28: Counting (i,j,k) with $1 \le i \le j \le k \le n$:
n is a positive integer. How many triples of integers from 1 through n can be formed where the elements of the triple are written in increasing order, but are not necessarily distinct?

P29: Counting iterations in a loop:
How many times will *stuff* be iterated:

```
for (i=1; i<= n; i++)
      for (j=1; j<= i; j++)
          for (k=1; k<=j; k++)
              /* do stuff */
```

P30: Number of integral solutions of an equation:
How many integer solutions to equation $x_1+x_2+x_3+ x_4 = 10$?

P31: Store problem:
- Bakery has balloons in 6 colors. Customer want to buy 20 balloons. How many different combinations?
- What if customer wants to buy at least 3 blue?
- What if shop has only 2 red balloons? (independent from 2nd question)